Gauge symmetries of systems with a finite number of degrees of freedom

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Abstract

For systems with a finite number of degrees of freedom, it is shown in [1] that first class constraints are Abelianizable if the Faddeev-Popov determinant is not vanishing for some choice of subsidiary constraints. Here, for irreducible first class constraint systems with SO(3) or SO(4) gauge symmetries, including a subset of coordinates in the fundamental representation of the gauge group, we explicitly determine the Abelianizable and non-Abelianizable classes of constraints. For the Abelianizable class, we explicitly solve the constraints to obtain the equivalent set of Abelian first class constraints. We show that for non-Abelianizable constraints there exist residual gauge symmetries which results in confinement-like phenomena.

1 Introduction

Gauge theories can be understood as constraint systems with first class constraints which are the generators of gauge transformation [2]. In the Dirac method of quantization, physical states are, by definition, invariant under gauge transformation. In gauge fixing approaches [3, 4, 5] like the Faddeev-Popov method [6], one eliminates the gauge freedom by introducing subsidiary constraints for which the Faddeev-Popov determinant is not vanish-

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ing. These methods are equivalent to Dirac quantization as they are believed to generate an equivalent set of physical observables.

Any given set of constraints $\{\phi_a\}$ can be replaced with a new set, say $\{\psi_a\}$, that is obtained by an invertible map from the original one. In this case, one says that $\{\phi_a\}$ and $\{\psi_a\}$ are *equivalent*. Usually, such a map is given as follows,

$$\psi_a = \sum_{b=1}^{A} C_{ab}\phi_b, \qquad a = 1, \cdots, A'. \tag{1}$$

where A and A' are the cardinality of the sets $\{\phi_a\}$ and $\{\psi_a\}$ respectively and C_{ab} are some functions of phase-space coordinates, which are not vanishing on the constraint surface. The set ϕ_a is *irreducible* if any equivalent set of constraints has the same cardinality, i.e. $A' = A$.

In the case of gauge theories with a finite number of degrees of freedom, it is known that an irreducible set of first class class constraints is *Abelianizable* if there exists a set of subsidiary constraints such that the Faddeev-Popov determinant is not vanishing [3, 1, 7]. By an Abelianizable set of constraints ϕ_a one means a set of constraints that is equivalent to a new set of constraints $\{\psi_a\}$ with the Poisson algebra $\{\psi_a, \psi_b\} = 0$.

Thus, in the case of non-Abelianizable first class constraints, which are the generators of gauge transformation in non-Abelian gauge theories, the Faddeev-Popov determinant is vanishing for any choice of gauge fixing conditions [1].

The proof is as follows: Consider a system with phase space coordinates z_μ, $\mu = 1, \cdots, 2N$, and a set of first class constraints ϕ_a, $a = 1, \cdots, A \leq N$ satisfying the algebra,

$$\{\phi_a, \phi_b\} = f_{abc}\phi_c, \tag{2}$$

where $\{\ ,\ \}$ stands for the Poisson bracket. Repeated indices are summed over. If ϕ_a's are non-Abelianizable, one can prove that $\left(\frac{\partial\phi_a}{\partial z_\mu}\right)$ is not full rank and consequently, as is stated above, the Faddeev-Popov determinant $\det(\{\phi_a, \omega_b\})$ is vanishing for any choice of subsidiary constraints ω_a, $a = 1, \cdots, A$ [1]. On the other hand if there exist a set of subsidiary constraints for which the Faddeev-Popov determinant is not vanishing, then one concludes that there exist a set of Abelian constraints equivalent to ϕ_a's [3, 1, 7]. The proof given in [3], is simple to follow: if for some set of subsidiary constraints the Faddeev-Popov determinant is not vanishing, then there exist

at least one set of A coordinates $\tilde{z}^a \in \{z^\mu\}$ for which $\det\left(\frac{\partial\phi^a}{\partial\tilde{z}^b}\right) \neq 0$. Thus one can solve the constraints $\phi^a = 0$ for $\tilde{z}$ to obtain a set of new *equivalent* constraints $\psi^a = \tilde{z}^a - f^a(z') = 0$, in which by z' one denotes the set of phase space coordinates complementary to $\tilde{z}$. It is now easy to show that ψ^a's are Abelian constraints. Indeed the Poisson brackets of new constraints with each other as given as follows,

$$\{\psi^a, \psi^b\} = \{\tilde{z}^a, \tilde{z}^b\} - \{\tilde{z}^a, f^b(z')\} - \{f^a(z'), \tilde{z}^b\} + \{f^a(z'), f^b(z')\}, \qquad (3)$$

is independent of $\tilde{z}$'s since $\{\tilde{z}^a, \tilde{z}^b\} = 0, \pm 1$. On the other hand the right hand side of Eq.(3) is vanishing on the constraint surface. Thus it vanishes identically [3].

Now consider a systems with a finite number of degrees of freedom with gauge group SO(N) including N coordinates, q_a, $a = 1, \cdots, N$ in the fundamental representation. The first class constraints for such systems has the following general from,

$$\phi_a = f_{abc} q^a p^b + L_a(q_i, p_i), \qquad (4)$$

where, f_{abc} is the structure constant of $\mathbf{so}(N)$ algebra, p_a's are momenta conjugate to q_a's and L_a are some functions of the other coordinates of systems and the corresponding momenta. Obviously, L_a's are generators of gauge transformation in the subspace of phase space spanned by q_i's and p_i's, i.e. $\{L_a, L_b\} = f_{abc} L_c$. Consequently the space of gauge orbits factorizes as $O_a \otimes O_i$.

Theorem $\mathbf{so}(N)$ *constraints are Abelianizable precisely if* $L_a \neq 0$ *for some* a*.*

Proof Assume $L_1 \neq 0$. The Faddeev-Popov determinant for the subsidiary constrains,

$$\omega_a = \begin{cases} q_a, & a = 1, \cdots, N-1, \\ p_1, & a = N, \end{cases} \qquad (5)$$

is not vanishing on the constraint surface,

$$\det\left(\{\omega_a, \phi_b\}\right) = -(q_N)^{N-2} L_1 \det f^{(N)}, \qquad (6)$$

where $f^{(N)}_{ab} = f_{Nab}$.

It is needless to say that the subsidiary constraints $q_a = 0$, $a = 1, \cdots, N$, for which the Faddeev-Popov determinant is vanishing are not *suitable* gauge

fixing conditions as they do not remove the gauge freedom. In fact, since the space of gauge orbits is factorized as $O_a \otimes O_i$ and the point $q_1 = \cdots = q_N = 0$ in O_a is stationary under SO(N) gauge transformation, in order to study gauge orbits and gauge transformations, one can concentrate on the space $O_a \backslash \{0\} \otimes O_i$. An open covering of the this space is given by open sets U_a in which $q_a \neq 0$. The subsidiary constraints (5) are gauge fixing conditions in the open set U_N.

Since the constraints ϕ_a are Abelianizable, it is interesting to solve them explicitly and obtain the equivalent set of Abelian constraints. For this, it suffices to find the Abelian constraints for U_N where $q_N \neq 0$. This appears to be a difficult task for general SO(N) gauge group, though formal solutions of such equations are given in [7].

In this paper, in sections 2 and 3, we calculate the explicit form of Abelian constraints for SO(3) and SO(4) cases respectively. In section 4, we study residual gauge symmetries in systems with non-Abelianizable first class constraints, and consider the discrete version of the Georgi-Glashow model [8] in which we obtain a simple confinement. Results are summarized in section 5. In appendix A, we study Abelianization of constraints in a discrete version of the Higgs sector of the standard model.

2 Abelianization of SO(3) constraints

Consider a system with **so**(3) gauge algebra given by the following first class constraints,

$$\begin{aligned} \phi_1 &= q_2 p_3 - q_3 p_2 + L_1, & (7) \\ \phi_2 &= q_3 p_1 - q_1 p_3 + L_2, & (8) \\ \phi_3 &= q_1 p_2 - q_2 p_1 + L_3. & (9) \end{aligned}$$

where L_a $(a = 1, 2, 3)$ is not a function of q_a's and p_a's.

If $q_3 \neq 0$, the constraints given in Eqs.(7-9) are equivalent to the following

constraints,

$$\psi_1 = \frac{\phi_1}{q_3} = p_2 - \frac{q_2}{q_3}p_3 - \frac{L_1}{q_3}, \tag{10}$$

$$\psi_2 = \frac{\phi_2}{q_3} = p_1 - \frac{q_1}{q_3}p_3 + \frac{L_2}{q_3}, \tag{11}$$

$$\psi_3 = \sum_{a=1}^{3} q_a\phi_a = q_1L_1 + q_2L_2 + q_3L_3. \tag{12}$$

It is easy to verify that $\{\psi_{1(2)}, \psi_3\} = 0$, and

$$\{\psi_1, \psi_2\} = -\frac{\psi_3}{(q_3)^3}. \tag{13}$$

To make the right hand side of Eq.(13) vanishing, ψ_2 can be redefined as follows,

$$\psi_2 \to \psi_2^{\rm new} = \psi_2^{\rm old} - \frac{q_2}{q_3}\frac{1}{(q_2)^2 + (q_3)^2}\psi_3. \tag{14}$$

it is easy to show that $\{\psi_{1(3)}, \psi_2^{\rm new}\} = 0$.

3 Abelianization of SO(4) constraints

Consider first class constraints in $\mathbf{so}(4)$ gauge algebra,

$$\phi_a = f_{abc}q_a p_c + L_a \tag{15}$$

where the non-vanishing structure coefficients are

$$f_{321} = f_{156} = f_{246} = f_{345} = 1, \tag{16}$$

we assume that $q_1 \neq 0$ and solve the constraints to obtain six equivalent Abelian constraints. First we replace ϕ_1 with ψ_1 defined as follows,

$$\psi_1 = \sum_{a=1}^{6} q_a\phi_a = \sum_{a=1}^{6} q_a L_a. \tag{17}$$

One can easily verify that the Poisson bracket of the other five constraints with ψ_1 is vanishing. Since $q_1 \neq 0$, one can solve constraints for p_a ($a =$

$2, 3, 5, 6)$ in terms of p_1 and p_4. It should be noted that by *solving constraints* one obtains a new set of constraints that are equivalent to the original ones in the sense of Eq.(1). Solving $\phi_2 = 0$ one obtains,

$$p_3 = \frac{q_3}{q_1}p_1 - \frac{q_4}{q_1}p_6 + \frac{q_6}{q_1}p_4 - \frac{L_2}{q_1}, \tag{18}$$

and $\phi_5 = 0$ gives,

$$p_6 = \frac{q_6}{q_1}p_1 - \frac{q_4}{q_1}p_3 + \frac{q_3}{q_1}p_4 + \frac{L_5}{q_1}. \tag{19}$$

Eqs.(18) and (19) can be solved to obtain,

$$\left(1 - \left(\frac{q_4}{q_1}\right)^2\right) p_3 = \left(\frac{q_3}{q_1} - \frac{q_4 q_6}{q_1^2}\right) p_1 + \left(\frac{q_6}{q_1} - \frac{q_4 q_3}{q_1^2}\right) p_4 - \left(\frac{L_2}{q_1} + \frac{q_4 L_5}{q_1^2}\right), \tag{20}$$

$$\left(1 - \left(\frac{q_4}{q_1}\right)^2\right) p_6 = \left(\frac{q_6}{q_1} - \frac{q_4 q_3}{q_1^2}\right) p_1 + \left(\frac{q_3}{q_1} - \frac{q_4 q_6}{q_1^2}\right) p_4 + \left(\frac{L_5}{q_1} + \frac{q_4 L_2}{q_1^2}\right). \tag{21}$$

For a generic point on the phase space, $q_1 \neq q_4$ and consequently constraints $\phi_2 = 0 = \phi_5$ are equivalent to the following new constraints,

$$\psi_3 = p_3 - \frac{q_3 q_1 - q_4 q_6}{q_1^2 - q_4^2}p_1 - \frac{q_6 q_1 - q_4 q_3}{q_1^2 - q_4^2}p_4 + \frac{q_1 L_2 + q_4 L_5}{q_1^2 - q_4^2}, \tag{22}$$

$$\psi_6 = p_6 - \frac{q_6 q_1 - q_4 q_3}{q_1^2 - q_4^2}p_1 - \frac{q_3 q_1 - q_4 q_6}{q_1^2 - q_4^2}p_4 - \frac{q_1 L_5 + q_4 L_2}{q_1^2 - q_4^2}. \tag{23}$$

Similarly one can show that constraints $\phi_3 = 0 = \phi_6$ are equivalent to the following constraints,

$$\psi_2 = p_2 - \frac{q_2 q_1 + q_4 q_5}{q_1^2 - q_4^2}p_1 + \frac{q_5 q_1 + q_4 q_2}{q_1^2 - q_4^2}p_4 - \frac{q_1 L_3 - q_4 L_6}{q_1^2 - q_4^2}, \tag{24}$$

$$\psi_5 = p_5 - \frac{q_5 q_1 + q_4 q_2}{q_1^2 - q_4^2}p_1 + \frac{q_2 q_1 + q_4 q_5}{q_1^2 - q_4^2}p_4 + \frac{q_1 L_6 - q_4 L_3}{q_1^2 - q_4^2}. \tag{25}$$

We define ψ_4 by solving ϕ_4 in terms of p_1 and p_4 using the above constraints,

$$\psi_4 = q_1 L_4 + q_4 L_1 - q_2 L_5 - q_5 L_2 + q_6 L_3 + q_3 L_6. \tag{26}$$

It is straightforward to show that $\{\psi_a, \psi_1\} = 0 = \{\psi_a, \psi_4\} = 0$, $(a = 1, \cdots, 6)$. Furthermore, one can show that,

$$\begin{aligned}
&\{\psi_2, \psi_5\} = \{\psi_3, \psi_6\} = 0, \\
&\{\psi_3, \psi_2\} = \{\psi_5, \psi_6\} = R_1\psi_1 - R_2\psi_4, \\
&\{\psi_2, \psi_6\} = \{\psi_3, \psi_5\} = R_2\psi_1 - R_1\psi_4,
\end{aligned} \tag{27}$$

where

$$R_1 = \frac{q_1^3 + 3q_1q_4^2}{\left(q_1^2 - q_4^2\right)^3}, \qquad R_2 = \frac{q_4^3 + 3q_4q_1^2}{\left(q_1^2 - q_4^2\right)^3}. \tag{28}$$

It is straightforward to show that by replacing ψ_3 and ψ_6 with the following equivalent constraints,

$$\begin{aligned} \psi_3 \rightarrow \psi_3^{\text{new}} &= \psi_3^{\text{old}} + S_1\psi_1 - S_2\psi_4, \\ \psi_6 \rightarrow \psi_6^{\text{new}} &= \psi_6^{\text{old}} - S_2\psi_1 + S_1\psi_4, \end{aligned} \tag{29}$$

$$\begin{aligned} S_1 &= \frac{1}{2}\left[\frac{(q_2+q_5)/(q_1-q_4)}{(q_1-q_4)^2+(q_2+q_5)^2} + \frac{(q_2-q_5)/(q_1+q_4)}{(q_1+q_4)^2+(q_2-q_5)^2}\right], \\ S_2 &= \frac{1}{2}\left[\frac{(q_2+q_5)/(q_1-q_4)}{(q_1-q_4)^2+(q_2+q_5)^2} - \frac{(q_2-q_5)/(q_1+q_4)}{(q_1+q_4)^2+(q_2-q_5)^2}\right]. \end{aligned} \tag{30}$$

one can obtain a new set of constraints which are Abelian.

3.1 Equivalence of Constraints

Two sets of first class constraints are equivalent precisely if the corresponding constraint surfaces and gauge transformations are equivalent. In the case studied here, the constraints surfaces of the SO(4) and the Abelian constraints are equivalent by construction, since the Abelian constraints are found by solving the SO(4) constraints. A possible flaw might be at the intersection of the constraint surface of ϕ_a given in Eq.(15) and the $q_1^2 - q_4^2 = 0$ surface. For example, Eqs.(18) and (19) imply that at $q_1^2 - q_4^2 = 0$,

$$p_3 + p_6 = \frac{q_3}{q_1}p_1 + \frac{q_6}{q_1}p_4 - \frac{L_2}{q_1}, \tag{31}$$

$$p_6 + p_3 = \frac{q_6}{q_1}p_1 + \frac{q_3}{q_1}p_4 + \frac{L_5}{q_1}. \tag{32}$$

Thus, it is necessary to see whether the constraints ψ_3 and ψ_6 given in Eqs.(22) and (23) give Eqs.(31) and (32) at $q_1^2 - q_4^2 = 0$. To deal with this problem, let's assume that for example, $q_1 = q_4 + \epsilon$ where ϵ is an infinitesimal parameter. By this assumption, Eqs.(22) and (23) give the following equations,

$$2\epsilon q_1 p_3 = \left(q_3q_1 - q_4q_6\right)p_1 + \left(q_6q_1 - q_4q_3\right)p_4 - \left(q_1L_2 + q_4L_5\right), \tag{33}$$

$$2\epsilon q_1 p_6 = \left(q_6q_1 - q_4q_3\right)p_1 + \left(q_3q_1 - q_4q_6\right)p_4 + \left(q_1L_5 + q_4L_2\right). \tag{34}$$

Eq.(33) for $\epsilon \to 0$ gives,

$$(q_3 - q_6)\, p_1 + (q_6 - q_3)\, p_4 - (L_2 + L_5) = 0 \tag{35}$$

Furthermore, by adding the left and right sides of Eqs.(33) and (34) one obtains,

$$2\epsilon q_1\, (p_3 + p_6) = \epsilon\left[(q_3 + q_6)\, p_1 + (q_6 + q_3)\, p_4 - (L_2 - L_5)\right]. \tag{36}$$

It is clear that Eqs.(36) and (35) give Eq.(31). Eq.(32) can be obtained in the same way and furthermore all these consistency checks can be done for the case $q_1 = -q_4 + \epsilon$ and $\epsilon \to 0$.

The above method of calculations motivates us to introduce an infinitesimal parameter in the denominators as follows,

$$\frac{1}{q_1^2 - q_4^2} \to \frac{1}{q_1^2 - q_4^2 + i\epsilon}, \tag{37}$$

and consider a rule for calculations: setting ϵ to zero is the final step in all calculation. The same rule resolves the ambiguity in the definition of functions S_1 and S_2 in Eq.(30).

To verify the equivalence of Abelian gauge transformations and the SO(4) gauge transformations, let θ_a and η_a $(a = 1, \cdots, 6)$ be the gauge parameters corresponding to the SO(4) and the Abelian gauge transformations respectively. θ_a and η_a are in general functions of phase space coordinates. The gauge transformation of a function of phase space coordinates like F is given as follows:

$$\delta^{\mathrm{A}} F = \sum_{a=1}^{6} \theta_a \{F, \phi_a\}, \tag{38}$$

$$\delta^{\mathrm{nA}} F = \sum_{a=1}^{6} \eta_a \{F, \psi_a\}. \tag{39}$$

The parameters $\eta_a = \delta^{\mathrm{A}} x_a$ $(a = 2, 3, 5, 6)$ can be determined in terms of the parameters θ_a by the condition $\delta^{\mathrm{A}} x_a = \delta^{\mathrm{nA}} x_a$ for $a = 2, 3, 5, 6$. A nontrivial observation is that these conditions give also $\delta^{\mathrm{A}} x_a = \delta^{\mathrm{nA}} x_a$ for $a = 1, 4$. This means that the Abelian and SO(4) gauge transformations of coordinates x_a are precisely equivalent. η_1 and η_4 can be determined after a lengthy but

quite straightforward calculation by examining the gauge transformation of, say, p_3.

In general, $\delta^{\mathrm{A}} F \approx \delta^{\mathrm{nA}} F$ where the symbol of *weak equality* $\approx$ means equality on the constraint surface [2]. This is good because for constraint systems, physical quantities are defined on the constraint surface. The non-trivial result above was the precise equivalence of the SO(4) and the Abelian gauge transformations for x_a's.

4 Residual $U(1)$ gauge symmetry

Now we deal with the class of systems with non-Abelianizable constraints. These are constraint systems for which $L_a = 0$ in Eq.(4).

Lemma *For constraints,*

$$\phi_a = f_{abc} q^a p^b. \tag{40}$$

there exist one non-gaugeable residual $U(1)$ *gauge symmetry generated by*

$$\psi_1 = \sum_a q_a \phi_a. \tag{41}$$

As we have seen, for $SO(4)$ constraints there exist another $U(1)$ residual gauge symmetry generated by ψ_4 given in Eq.(26).

The existence of residual gauge symmetries in systems with non-Abelianizable first class constraints is a consequence of the main theorem in [1] as can be seen as follows.

Corollary *In a gauge theory with non-Abelianizable gauge symmetry, any classical configuration* $z_\mu = z_\mu^v$ *is invariant under a non-trivial subgroup of the non-Abelian gauge group.*

To see this, let's define the generators of gauge transformation by

$$\delta_a \eta(z_\mu) = \{\eta, \phi_a\} \tag{42}$$

in which $\eta(z_\mu)$ is any function of the phase space coordinates. Define λ_i^a, $i = 1, \cdots, I$ to be the i-th null vector of $\left(\frac{\partial \phi_a}{\partial z_\mu}\right)_{z_\mu^v}$, and define $\delta_i = \lambda_i^a \delta_a$. Now it is easy to verify that $(\delta_i \eta)_{z^v} = 0$ for any function $\eta(z)$. We rearrange the A generators of gauge transformation to I δ_i's and the complementary set δ_α where the index α runs over $1, \cdots, A-I$. One can consider $A-I$ subsidiary

constraints which gauge the gauge freedom corresponding to δ_α's. But since $\delta_i \eta|_{z^v} = 0$ for any function η, there is no way to gauge the gauge symmetry generated by δ_i's. Recall that a gauge fixing condition is a function ω which is not invariant under the gauge transformations. Consequently, any classical configuration of the system is invariant under the gauge transformation generated by δ_i's.

4.1 Quantization

To deal with δ_i's, the only consistent method of quantization is to use the Dirac definition of physical states. Thus after imposing the $A - I$ possible gauge fixing conditions to gauge δ_α's, one defines/assumes the physical state to be invariant under H_i's which are the quantum operators corresponding to the classical generators δ_i's. It is a natural assumption since δ_i's are by definition the symmetries of the classical configurations. This implies that the only physical observables are those combinations of field operators that are invariant under H_i's. This is again in agreement with the classical result $\delta_i \eta|_{z^v} = 0$. This phenomena can be interpreted as confinement.

We state without proof the following conjecture.

Conjecture: *H_i's are the generators of the Cartan subalgebra of the gauge group G generated by ϕ_a's and consequently I is equal to the rank of the gauge group.*

If this conjecture is valid then one verifies that the number of gauge symmetries that can be fixed by gauge fixing conditions equals the number of non zero roots of the gauge group.

4.2 Example

Here we give an illustrative simple examples which shed some light on different aspects of the general arguments and statements given above.

This is the discrete version of the Georgi-Glashow model in which we obtain a simple confinement. The model is given by the Lagrangian $L = \frac{1}{2}\dot{\vec{q}}^2 - V(\vec{q})$, which is invariant under the action of $SO(3)$ [8]:

$$q \to gq, \qquad g = e^{i\theta \hat{n}.\phi}, \ \phi \in so(3). \tag{43}$$

For example, $V(\vec{q}) = (\vec{q}^2 - a^2)^2$. It is easy to verify that any classical vacuum $\vec{q}^v$ ($|\vec{q}^v| = a$), is invariant under the $U(1)$ subgroup of $SO(3)$ generated by $\hat{q}^v.\phi$ as is expected. In fact $\delta_{\epsilon,\hat{n}} q_i^v = \epsilon(\hat{n} \times \vec{q}^v)_i$ which is vanishing if $\hat{n} = \hat{q}^v$.

To make connection between this seemingly trivial result and the general arguments given above, let's consider a gauge field $\mathbf{A}$ in the adjoint representation of $so(3)$ and the Lagrangian $L = \frac{1}{2}(D_t\vec{q})^2 - V(\vec{q})$, where $D_t\vec{q} = \dot{\vec{q}} + \mathbf{A}\vec{q}$. The corresponding Hamiltonian is $H = H_0 + A_i L_i$ where $A_i = \epsilon_{ijk}\mathbf{A}_{ij}$, $\phi_i = \epsilon_{ijk}q_j p_k$ and $H_0 = \frac{1}{2}\vec{p}^2 + V(\vec{q})$. $\vec{p} = \dot{\vec{q}} + \mathbf{A}\vec{q}$ is the momentum conjugate to $\vec{q}$. The momenta conjugate to the gauge field A are vanishing. These are the primary first class constraints. The corresponding equations of motion result in the secondary first class constraints $\phi_i = 0$. The secondary constraints here are the generators of the gauge group $SO(3)$ as they satisfy the algebra, $\{\phi_i, \phi_j\} = \epsilon_{ijk}\phi_k$. One can easily show that $\hat{q}^v$ is the unique null vector of

$$\left(\frac{\partial\phi_i}{\partial z_\mu}\right)_{z^v_\mu} = \begin{pmatrix} 0 & 0 & 0 & 0 & -q^v_3 & q^v_2 \\ 0 & 0 & 0 & q^v_3 & 0 & -q^v_1 \\ 0 & 0 & 0 & -q^v_2 & q^v_1 & 0 \end{pmatrix}, \tag{44}$$

where $z_\mu = (\vec{q}, \vec{p})$ are the phase space coordinates and $z^v_\mu = (\vec{q}^v, \vec{0})$ is a classical vacuum state. Therefore the vacuum $\vec{q}^v$ is invariant under $U(1) \subset SO(3)$ generated by $\hat{q}^v.L$.

What is the confinement in this example? We argued that in general the maximum number of gauge degrees of freedom that can be gauged equals rank $\left(\frac{\partial\phi_a}{\partial z_\mu}\right) = A - I$ in which A is the number of the gauge generators. From Eq.(44) one verifies that $I = 1$ and consequently one can impose at most two gauge fixing conditions. Let's assume that these two subsidiary constraints are $q_3 = 0 = p_3$. Namely we are assuming that the trajectory of the particle is in the 1-2 plane. Since the total angular momentum is vanishing the trajectory is a straight line which can be assumed to pass through the origin without loss of generality. The $U(1)$ symmetry here is the symmetry under arbitrary rotation of this line around the third axis. Let's define new coordinates $z = q_1 + iq_2$ and $\bar{z} = q_1 + iq_2$, which under the $U(1)$ transformation change a phase, $z \to e^{-i\wp}z$ and $\bar{z} \to e^{i\wp}\bar{z}$. Assuming that the vacuum state $|0\rangle$ corresponding to the classical vacua $\vec{q}^v = (0, 0, q^v)$ is invariant under symmetries of the classical vacua, one verifies that e.g. $\langle z\rangle = 0$ while $\langle z\bar{z}\rangle$ can be in general non-vanishing. Considering z as a *quark*, this observation can be interpreted as confinement.

5 Summary

For first class constraint systems with first class constraints

$$\phi_a = f_{abc} q^a p^b + L_a(q_i, p_i), \tag{45}$$

satisfying constraint algebra,

$$\{\phi_a, \phi_b\} = f_{abc}\phi_c, \tag{46}$$

in which f_{abc} is the structure coefficients of SO(3) or SO(4) Lie algebras, and at least one $L_a \neq 0$, we obtained the equivalent set of Abelian first class constraints.

For **so**(3) gauge algebra, with structure coefficient $f_{abc} = \epsilon_{abc}$ the Abelian constraints for the $q_3 \neq 0$ subset of phase space are given as follows,

$$\begin{aligned}
\psi_1 &= p_2 - \frac{q_2}{q_3}p_3 - \frac{L_1}{q_3} \\
\psi_2 &= p_1 - \frac{q_1}{q_3}p_3 + \frac{L_2}{q_3} - \frac{q_2}{q_3}\frac{1}{(q_2)^2 + (q_3)^2}\psi_3 \\
\psi_3 &= q_1 L_1 + q_2 L_2 + q_3 L_3.
\end{aligned} \tag{47}$$

Appropriate transition functions will give the corresponding Abelian constraints in the $q_1 \neq 0$ and $q_2 \neq 0$ subsets of the phase space. We have excluded the point $q_1 = q_2 = q_3 = 0$, which is stationary under gauge transformations.

For **so**(4) gauge algebra, with non-vanishing structure coefficients

$$f_{321} = f_{156} = f_{246} = f_{345} = 1, \tag{48}$$

the Abelian constraints in the $q_1 \neq 0$ subset of phase space are,

$$\begin{aligned}
&\psi_1 = \textstyle\sum_{a=1}^{6} q_a L_a, \\
&\psi_2 = p_2 - \tfrac{q_2 q_1 + q_4 q_5}{q_1^2 - q_4^2 + i\epsilon} p_1 + \tfrac{q_5 q_1 + q_4 q_2}{q_1^2 - q_4^2 + i\epsilon} p_4 - \tfrac{q_1 L_3 - q_4 L_6}{q_1^2 - q_4^2 + i\epsilon}, \\
&\psi_3 = p_3 - \tfrac{q_3 q_1 - q_4 q_6}{q_1^2 - q_4^2 + i\epsilon} p_1 - \tfrac{q_6 q_1 - q_4 q_3}{q_1^2 - q_4^2 + i\epsilon} p_4 + \tfrac{q_1 L_2 + q_4 L_5}{q_1^2 - q_4^2 + i\epsilon} + S_1\psi_1 - S_2\psi_4, \\
&\psi_4 = q_1 L_4 + q_4 L_1 - q_2 L_5 - q_5 L_2 + q_6 L_3 + q_3 L_6, \\
&\psi_5 = p_5 - \tfrac{q_5 q_1 + q_4 q_2}{q_1^2 - q_4^2} p_1 + \tfrac{q_2 q_1 + q_4 q_5}{q_1^2 - q_4^2 + i\epsilon} p_4 + \tfrac{q_1 L_6 - q_4 L_3}{q_1^2 - q_4^2 + i\epsilon}, \\
&\psi_6 = p_6 - \tfrac{q_6 q_1 - q_4 q_3}{q_1^2 - q_4^2 + i\epsilon} p_1 - \tfrac{q_3 q_1 - q_4 q_6}{q_1^2 - q_4^2 + i\epsilon} p_4 - \tfrac{q_1 L_5 + q_4 L_2}{q_1^2 - q_4^2 + i\epsilon} - S_2\psi_1 + S_1\psi_4,
\end{aligned} \tag{49}$$

where S_1 and S_2 are defined in Eq.(30) and ϵ is a parameter which one sets to zero at the end of calculations. This parameter is introduced to resolve the apparent singularity at $q_1^2 - q_4^2 = 0$.

For the non-Abelianizable constraints, which is the case with $L_a = 0$ in Eq.(45), there are exist residual gauge symmetries which results in confinement-like phenomena.

First class constraint systems with SO(N) gauge symmetry generated by first class class constraints (45) are interesting specially as toy models to study Gribov copies in non-Abelian gauge theories. Results given in Eqs.(47) and (49) can be used to study this problem from a new point of view.

A The Higgs sector of the standard model

In this appendix, we study a special SO(4) invariant constraint system in which the **so**(4) Lie algebra is represented by first class constraints constraints in a different way in comparison to section 3.

We consider a system with 4 degrees of freedom q_α, plus a gauge field. The index α runs over 0,1,2,3. The Lagrangian is the following $L = 1/2(\dot{q}_\alpha - A_i \eta^i_{\alpha\beta} q_\beta)^2 - V(q)$ where Latin indices i runs over 1,2,3. Repeated indices are summed over. The potential can be taken to be $V(q) = \lambda(q_\alpha^2 - 1)^2$. The symbols $\eta^i_{\alpha\beta}$ are 't Hooft symbols,

$$\eta^i_{\alpha\beta} = \epsilon_{0i\alpha\beta} - \delta_{i\alpha}\delta_{0\beta} + \delta_{0\alpha}\delta_{i\beta}, \tag{50}$$

satisfying the commutation relation, $\eta^{[i}_{\alpha_\rho}\eta^{j]}_{\beta_\rho} = 2\epsilon_{ijk}\eta^k_{\alpha\beta}$. This looks just like a discrete version of the Higgs sector of the standard model.

The conjugate momentum to the gauge field A_i vanishes. These are the primary constraints. The secondary first class constraints are obtained by differentiation with respect to A_i. They are $\phi_i = -p_\alpha \eta^i_{\alpha\beta} q_\beta$ where $p_\alpha = (\dot{q}_\alpha - A_i \eta^i_{\alpha\beta} q_\beta)$ is the conjugate momentum to q_α. It is easy to see, using the expression of the 't Hooft symbols that $\{\phi_i, \phi_j\} = 2\epsilon_{ijk}\phi_k$. Thus ϕ_i are non-Abelian constraints generating a SU(2) subgroup of SO(4). Now one can introduce the following subsidiary constraints, which are equivalent to unitary gauge $q_i = 0$. The Poisson brackets of these constraints with the ϕ_i are $\{q_i, \phi_j\} = q_0 \delta_{ij}$ which is non-vanishing for $q^0 \neq 0$.

We show that ϕ_i's are Abelianizable if $q_0 \neq 0$. Thus we are realizing two different sectors in the theory. In one sector q_0 and consequently p_0 are both

vanishing as we will show in a moment. Thus the SO(4) model reduces to the SO(3) model studied in section 2. In the sector $q_0 \neq 0$, we show that the secondary constraints $\phi_i = 0$ are equivalent to three Abelian constraints.

The proof is as follows. Using Eq.(50), one can show that,

$$\vec{\phi} = \vec{q} \times \vec{p} + q_o \vec{p} - p_0 \vec{q}, \tag{51}$$

in which $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$. The constraint $\vec{\phi} = 0$ implies that

$$\begin{aligned} \vec{q}.\vec{\phi} &= q_0 \vec{q}.\vec{p} - p_0 \vec{q}^2 = 0, \\ \vec{p}.\vec{\phi} &= p_0 \vec{q}.\vec{p} - q_0 \vec{p}^2 = 0. \end{aligned} \tag{52}$$

The cases with vanishing $\vec{q}^2$ or $\vec{p}^2$ are rather trivial. The most nontrivial cases are given either by $q_0 = p_0 = 0$ or by $\vec{\psi} = q_0 \vec{p} - p_0 \vec{q} = 0$ and $q_0 \neq 0 \neq p_0$. In the first case, one obtains the SO(3) model and $\det(\{q_i, \phi_j\}) = 0$ whatever the gauge fixing conditions are. In the second case, the constraints $\phi_i = 0$ are Abelianizable as they are equivalent to Abelian constraints $\psi_i = 0$. ϕ_i's and ψ_i's are equivalent as they define the same constraint surface in the phase space. But by "equivalence" in [1] one means also equivalence in the gauge transformation generated by two sets of first class constraints which we have not verified yet. The gauge transformation generated by ϕ_i's is given by $\delta \vec{q} = \{\vec{q}, \vec{n}.\vec{\phi}\} = \vec{n} \times \vec{q} + q_0 \vec{n}$, where $\vec{n}$ is the parameter of gauge transformation. Since $q_0 \neq 0$ one can easily verify that $\delta \vec{q} = \{\vec{q}, \vec{n}'.\vec{\psi}\}$ in which $\vec{n}' = \vec{n} + q_0^{-1} \vec{n} \times \vec{q}$.

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